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Affine Kac-Moody Algebras and Polydifferentials

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Let $\mathfrak{g}$ be a finite-dimensional algebra over $\mathbb{C}$ with a fixed invariant symmetric bilinear form $c(x, y)$. (Readers accustomed to traditional notation may assume that $c(x, y) = c \cdot (x, y)$ where (x, y) is an invariant scalar product on $\mathfrak{g}$ and $c \in \mathbb{C}$ is the "level".) Set $\mathcal{O} = \mathbb{C}[[z]]$, $K = \mathbb{C}((z))$, $\mathfrak{m} = z\mathbb{C}[[z]] \subset \mathcal{O}$, and $\mathfrak{g}_K = \mathfrak{g} \otimes K$. We consider $\mathfrak{g}_K$ as a Lie algebra over $\mathbb{C}$. The 2-cocycle $B : \mathfrak{g}_K \times \mathfrak{g}_K \to \mathbb{C}$ given by

$$B(u, v) = \operatorname*{res}_{z=0} c(u'(z), v(z))dz \tag{1}$$

defines a central extension of $\mathfrak{g}_K$ which will be denoted $\hat{\mathfrak{g}}_K^c$. As a vector space $\hat{\mathfrak{g}}_K^c$ is the direct sum of $\mathfrak{g}_K$ and a one-dimensional vector space generated by an element $\imath$. The commutator in $\hat{\mathfrak{g}}_K^c$ is denoted by $[\,\cdot\,,\,\cdot\,]_c$ and defined by

$$[u, v]_c = [u, v] + B(u, v) \cdot \imath \qquad \text{for } u, v \in \mathfrak{g}_K \tag{2a}$$

$$[\imath, u] = 0 \qquad \text{for } u \in \hat{\mathfrak{g}}_K^c. \tag{2b}$$

Set $U^c\mathfrak{g}_K = U\hat{\mathfrak{g}}_K^c/(\imath - 1)$ where $U\hat{\mathfrak{g}}_K^c$ is the universal enveloping algebra of $\hat{\mathfrak{g}}_K^c$. Usually we will write U^c instead of $U^c\mathfrak{g}_K$. The standard filtration of $U\hat{\mathfrak{g}}_K^c$ induces a filtration $U^c_\bullet$ on U^c, where U^c_k is the vector space generated by products of $\leq k$ elements of $\mathfrak{g}_K$. Let I^c_n be the left ideal of U^c generated by $\mathfrak{g} \otimes \mathfrak{m}^n \subset \mathfrak{g}_K, n \geq 0$. Set $I^c_{n,k} = I^c_n \cap U^c_k$. Using the Poincaré-Birkhoff-Witt theorem and the fact that $\mathfrak{g} \otimes \mathfrak{m}^n$ is a subalgebra of $\hat{\mathfrak{g}}_K^c$ it is easy to show that

$$I^c_{n,k} = U^c_{k-1} \cdot (\mathfrak{g} \otimes \mathfrak{m}^n). \tag{3}$$

If $c = 0$ then $U^c\mathfrak{g}_K = U\mathfrak{g}_K$. In this case we write $U_k, I_n, I_{n,k}$ instead of $U^c_k, I^c_n, I^c_{n,k}$.

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In representation theory one usually considers the completion

$$\widehat{U}^c := \varinjlim_k \varprojlim_n U^c_k/I^c_{n,k}$$

rather than U^c itself. The goal of this paper is to describe the dual space $(\widehat{U}^c)^*$. In order to do this it is enough to describe the space $(U^c_k/I_{n,k})^*$ for all n and k and the natural mappings between them. This is easy for $k = 1$. Indeed, since $U^c_1/I^c_{n,1} = \mathbb{C} \oplus (\mathfrak{g} \otimes (K/\mathfrak{m}^n))$, we have $(U^c_1/I^c_{n,1})^* = \mathbb{C} \oplus (\mathfrak{g}^* \otimes (K/\mathfrak{m}^n)^*)$ and $(K/\mathfrak{m}^n)^*$ can be identified with the space of differentials $w = f(z)dz$ where $f \in z^{-n}\mathbb{C}[[z]] \subset K$. (A differential w defines a linear functional $\varphi \to \operatorname{res}_{z=0} \varphi w$, $\varphi \in K$.)

If $k > 1$ we need some notation. Let $\Omega_{\mathcal{O}}$ be the module of continuous differentials of $\mathcal{O}$; it consists of expressions $f(z)dz$, $f \in \mathcal{O}$. Set $\Omega_K = \Omega_{\mathcal{O}} \otimes_{\mathcal{O}} K$. Denote by $\mathcal{O}_r$ (resp. $\Omega^{\mathcal{O}}_r$) the completed tensor product of r copies of $\mathcal{O}$ (resp. of $\Omega_{\mathcal{O}}$). Set $\Omega^K_r = \Omega^{\mathcal{O}}_r \otimes_{\mathcal{O}_r} K_r$ where K_r is the field of fractions of $\mathcal{O}_r$. We identify $\mathcal{O}_r$ with $\mathbb{C}[[z_1, \ldots, z_r]]$ and write elements of Ω^K_r as $f(z_1, \ldots, z_r)dz_1 \ldots dz_r$ where f belongs to the field of fractions of $\mathbb{C}[[z_1, \ldots, z_r]]$. Elements of Ω^K_r will be called *polydifferentials*. The only difference between polydifferentials and differential r-forms is that an element σ of the symmetric group S_r is supposed to map $f(z_1, \ldots, z_r)dz_1 \cdots dz_r$ to $f(z_{\sigma(1)}, \ldots, z_{\sigma(r)})dz_1 \cdots dz_r$ while $\sigma(f(z_1, \ldots, z_r)dz_1 \wedge \cdots \wedge dz_r) = (-1)^{l(\sigma)} f(z_{\sigma(1)}, \ldots, z_{\sigma(r)})dz_1 \wedge \cdots \wedge dz_r$ where $l(\sigma)$ is the number of inversions.

We are going to construct a canonical isomorphism between $(U^c_k/I^c_{n,k})^*$ and the following space $\Omega^c_{n,k}$.

Definition. $\Omega^c_{n,k}$ is the set of $(k+1)$-tuples $(w_0, \ldots, w_k)$, $w_r \in (\mathfrak{g}^*)^{\otimes r} \otimes \Omega^K_r$, such that:

(1) w_r is invariant with respect to the action of the symmetric group S_r (S_r acts both on $(\mathfrak{g}^*)^{\otimes r}$ and on Ω^K_r);

(2) w_r has poles of order $\leq n$ at the hyperplanes $z_i = 0$, $1 \leq i \leq r$, poles of order ≤ 2 at the hyperplanes $z_i = z_j$, $1 \leq i < j \leq r$, and no other poles;

(3) if $w_r(z_1, \ldots, z_r) = f_r(z_1, \ldots, z_r)dz_1 \cdots dz_r$, $r \geq 2$, then

$$f_r(z_1, \ldots, z_r) = \frac{f_{r-2}(z_1, \ldots, z_{r-2}) \otimes c}{(z_{r-1} - z_r)^2} + \frac{\varphi^*\left(f_{r-1}(z_1, \ldots, z_{r-1})\right)}{z_{r-1} - z_r} + \cdots. \tag{4}$$

Here c is considered as an element of $\mathfrak{g}^* \otimes \mathfrak{g}^*$, $\varphi^* : (\mathfrak{g}^*)^{\otimes(r-1)} \to (\mathfrak{g}^*)^{\otimes r}$ is dual to the mapping $\varphi : \mathfrak{g}^{\otimes r} \to \mathfrak{g}^{\otimes(r-1)}$ given by $\varphi(a_1 \otimes \cdots \otimes a_r) = a_1 \otimes \cdots \otimes a_{r-2} \otimes [a_{r-1}, a_r]$, and the dots in (4) denote an expression which does not have a pole at the generic point of the hyperplane $z_{r-1} = z_r$.

Let us explain that in (4) we consider f_r as a function with values in $(\mathfrak{g}^*)^{\otimes r}$.

The space $\Omega^c_{n,k}$ is equipped with the topology induced by the embedding $\Omega^c_{n,k} \hookrightarrow \prod_{0 \leq r \leq k} (\mathfrak{g}^*)^{\otimes r} \otimes \Omega^{\mathcal{O}}_r$ given by $(w_0, \ldots, w_k) \mapsto (\eta_0, \ldots, \eta_k)$, $\eta_r = \prod_i z_i^n \cdot \prod_{i<j} (z_i - z_j)^2 \cdot w_r$.

We will denote $\Omega^c_{n,k}$ for $c = 0$ simply by $\Omega_{n,k}$.

Theorem. (1) There is a pairing $\langle , \rangle : U^c_k \times \Omega^c_{n,k} \to \mathbb{C}$ such that, if $0 \le r \le k$, $u_1, \dots, u_r \in \mathfrak{g}_K$, $w = (w_0, \dots, w_k) \in \Omega^c_{n,k}$, then

$$\langle u_1 \cdots u_r, w\rangle = \operatorname*{res}_{z_1=0} \cdots \operatorname*{res}_{z_r=0} \big(u_1(z_1) \otimes \cdots \otimes u_r(z_r), w_r(z_1, \dots, z_r)\big) . \tag{5}$$

(2) The pairing (5) defines a topological isomorphism $\Omega^c_{n,k} \xrightarrow{\sim} (U^c_k/I^c_{n,k})^*$ where the topology on $U^c_k/I^c_{n,k}$ is assumed to be discrete.

Let us explain that in (5) $u_1(z_1) \otimes \cdots \otimes u_r(z_r)$ is a function with values in $\mathfrak{g}^{\otimes r}$, w_r is a polydifferential with values in $(\mathfrak{g}^*)^{\otimes r}$, $(u_1(z_1) \otimes \cdots \otimes u_r(z_r),\ w_r(z_1, \dots, z_r))$ is a scalar-valued polydifferential, and the notation $\operatorname{res}_{z_1=0} \cdots \operatorname{res}_{z_r=0}$ means that we first compute the residue with respect to z_r considering $z_1, \dots, z_{r-1}$ as parameters, and then we compute the residue with respect to z_{r-1}, etc. For instance, to compute

$$\operatorname*{res}_{z_1=0} \operatorname*{res}_{z_2=0} \psi(z_1, z_2) dz_1\, dz_2,$$

we have to consider ψ as an element of $\mathbb{C}((z_1))((z_2))$ and find the coefficient $a_{-1,-1}$ in the corresponding power series

$$\psi(z_1, z_2) = \sum_{j=-m}^{\infty} z_2^j \cdot \sum_{i=-N(m)}^{\infty} a_{ij}\, z_1^i. \tag{6}$$

By abuse of language, (6) will be called the power series decomposition of ψ in the domain $|z_1| \gg |z_2|$. Notice that if $\psi(z_1, z_2)$ is meromorphic, then

$$\operatorname*{res}_{z_1=0} \operatorname*{res}_{z_2=0} \psi(z_1, z_2) dz_1\, dz_2 = \frac{1}{(2\pi i)^2} \oint_{|z_1|=\varepsilon_1} \oint_{|z_2|=\varepsilon_2} \psi(z_1, z_2) dz_1\, dz_2, \qquad 1 \gg \varepsilon_1 \gg \varepsilon_2 > 0.$$

Proof of the theorem. To prove statement (1) we have to show that

$$\begin{aligned}
&\operatorname*{res}_{z_1=0} \cdots \operatorname*{res}_{z_r=0} \big(u_1(z_1) \otimes \cdots \otimes u_{i-1}(z_{i-1}) \otimes (u_i(z_i) \otimes u_{i+1}(z_{i+1}) \\
&\qquad - u_{i+1}(z_i) \otimes u_i(z_{i+1})) \otimes u_{i+2}(z_{i+2}) \otimes \cdots \otimes u_r(z_r), w_r(z_1, \dots, z_r)\big) \\
&\quad = \operatorname*{res}_{z_1=0} \cdots \operatorname*{res}_{z_{r-1}=0} (u_1(z_1) \otimes \cdots \otimes u_{i-1}(z_{i-1}) \otimes [u_i(z_i), u_{i+1}(z_i)] \\
&\qquad \otimes u_{i+2}(z_{i+1}) \otimes \cdots \otimes u_r(z_{r-1}), w_{r-1}(z_1, \dots, z_{r-1})) \\
&\qquad + \operatorname*{res}_{z=0} c(u_1'(z), u_2(z)) dz \cdot \operatorname*{res}_{z_1=0} \cdots \operatorname*{res}_{z_{r-2}=0} (u_1(z_1) \otimes \cdots \otimes u_{i-1}(z_{i-1}) \\
&\qquad \otimes u_{i+2}(z_i) \otimes \cdots \otimes u_r(z_{r-2}), w_{r-2}(z_1, \dots, z_{r-2})).
\end{aligned} \tag{7}$$

Since w_r is S_r-invariant, the left-hand side of (7) can be rewritten as

$$\operatorname*{res}_{z_1=0}\cdots\operatorname*{res}_{z_{i-1}=0}\operatorname*{res}_{z_i=0}\operatorname*{res}_{z_{i+1}=0}\eta(z_1,\ldots,z_{i+1}) - \operatorname*{res}_{z_1=0}\ldots\operatorname*{res}_{z_{i-1}=0}\operatorname*{res}_{z_{i+1}=0}\operatorname*{res}_{z_i=0}\eta(z_1,\ldots,z_{i+1}) \tag{8}$$

where $\eta(z_1,\ldots,z_{i+1}) = \operatorname{res}_{z_{i+2}=0}\cdots\operatorname{res}_{z_r=0}(u_1(z_1)\otimes\cdots\otimes u_r(z_r), w_r(z_1,\ldots,z_r))$. η has poles only at the hyperplanes $z_k=0, 1\le k\le i+1$, and $z_k=z_l, 1\le k<l\le i+1$. Therefore

$$\operatorname*{res}_{z_i=0}\operatorname*{res}_{z_{i+1}=0}\eta(z_1,\ldots,z_{i+1}) - \operatorname*{res}_{z_{i+1}=0}\operatorname*{res}_{z_i=0}\eta(z_1,\ldots,z_{i+1}) = -\operatorname*{res}_{z_i=0}\operatorname*{res}_{z_{i+1}=z_i}\eta(z_1,\ldots,z_{i+1}) \tag{9}$$

where the right-hand side is understood as follows: we first consider z_i as a parameter, compute the residue at $z_{i+1}=z_i$, and then compute the residue at $z_i=0$. Let us explain that if η is meromorphic, (9) is easily obtained by expressing residues as Cauchy integrals, while in the general case one can either prove (9) by direct computations or deduce it from *Parshin's residue formula* [P, §1, Proposition 7] which asserts that, if f belongs to the field of fractions of $\mathbb{C}[[z,u]]$, then

$$\sum_C \operatorname*{res}_{z=u=0}\operatorname*{res}_{C} f(z,u)dz\wedge du = 0 \tag{10}$$

where the summation is over all irreducible "formal curves" $\varphi(z,u)=0, \varphi\in\mathbb{C}[[z,u]]$.

It is easy to deduce from (4) that

$$\begin{aligned}\operatorname*{res}_{z_{i+1}=z_i}\eta(z_1,\ldots,z_{i+1}) = {} & c(u_i(z_i),u'_{i+1}(z_i))\cdot\operatorname*{res}_{z_{i+2}=0}\cdots\operatorname*{res}_{z_r=0}(u_1(z_1)\\ & \otimes\cdots\otimes u_{i-1}(z_{i-1})\otimes u_{i+2}(z_{i+2})\otimes\cdots\\ & \otimes u_r(z_r), w_{r-2}(z_1,\ldots,z_{i-1},z_{i+2},\ldots,z_r))\\ & -\operatorname*{res}_{z_{i+1}=0}\cdots\operatorname*{res}_{z_{r-1}=0}(u_1(z_1)\otimes\cdots\otimes u_{i-1}(z_{i-1})\\ & \otimes[u_i(z_i),u_{i+1}(z_i)]\otimes u_{i+2}(z_{i+1})\otimes\cdots\\ & \otimes u_r(z_{r-1}), w_{r-1}(z_1,\ldots,z_{r-1})).\end{aligned} \tag{11}$$

It follows from (9) and (11) that (8) is equal to the right-hand side of (7). So we have proved the statement (1) of the theorem.

Since the order of the pole of w_r at $z_r=0$ is $\le n$, the right-hand side of (5) vanishes provided $u_r\in\mathfrak{g}\otimes\mathfrak{m}^n$. Taking into account (3), we see that the pairing (5) defines a mapping $\psi:\Omega^c_{n,k}\to(U^c_k/I^c_{n,k})^*$. If $(w_0,\ldots,w_k)\in\Omega^c_{n,k}$ and $\lambda=\psi(w_0,\ldots,w_k)$, then (5) shows that

$w_r(z_1,\dots,z_r)$ has the following power series decomposition in the domain $|z_1| \gg |z_2| \gg \cdots \gg |z_r|$:

$$w_r(z_1,\dots,z_r) = \sum_{l_1,\dots,l_r} \sum_{i_1,\dots,i_r} \lambda(e_{i_1}^{(l_1)} \cdots e_{i_r}^{(l_r)}) e^{i_1} \otimes \cdots \otimes e^{i_r} \times z_1^{-l_1-1} \cdots z_r^{-l_r-1} dz_1 \cdots dz_r. \tag{12}$$

Here $\{e^i\}$ is a basis of $\mathfrak{g}^*$ and $e_i^{(l)} = e_i z^l \in \mathfrak{g}_K \subset U^c$ where e_i is the dual basis of $\mathfrak{g}$. (12) implies that ψ is injective. To prove the surjectivity of ψ we must show that for any $\lambda \in (U_k^c/I_{n,k}^c)^*$ the $(k+1)$-tuple $(w_0,\dots,w_k)$ defined by (12) belongs to $\Omega_{n,k}^c$. Clearly, $w_r = f_r dz_1 \cdots dz_r$ where $f_r \in (\mathfrak{g}^*)^{\otimes r} \otimes \mathbb{C}((z_1)) \cdots ((z_r))$. We must first of all prove that $f_r \in (\mathfrak{g}^*)^{\otimes r} \otimes K_r$ where K_r is the field of fractions of $\mathbb{C}[[z_1,\dots,z_r]]$. (This may be considered as a kind of analytical continuation of the right-hand side of (12).) We also have to verify the properties (1)–(3) from the definition of $\Omega_{n,k}^c$.

Let us introduce the "fields" $A_i(\zeta)$ defined by

$$A_i(\zeta) = \sum_l e_i^{(l)} \zeta^{-l-1} \tag{13}$$

where $e_i^{(l)}$ has the same meaning as in (12). (Since $e_i^{(l)} = e_i z^l$ we can write heuristically $A_i(\zeta) = \delta(z-\zeta) e_i$ where the "δ-function" is defined by $\delta(z-\zeta) = \sum z^l \zeta^{-l-1}$.) $A_i(\zeta)$ is a formal power series in ζ with coefficients in $\mathfrak{g}_K \subset U^c$. Now we can rewrite (12) as

$$w_r(z_1,\dots,z_r) = \sum_{i_1,\dots,i_r} \lambda(A_{i_1}(z_1) \cdots A_{i_r}(z_r)) \cdot (e^{i_1} \otimes \cdots \otimes e^{i_r}) dz_1 \cdots dz_r. \tag{14}$$

Since $[e_i^{(l)}, e_j^{(m)}] = l c_{ij}\, \delta_{l,-m} + \sum f_{ij}^q\, e_q^{(l+m)}$, where c_{ij} is the matrix of the bilinear form c and f_{ij}^q are the structure constants of $\mathfrak{g}$, we have $[A_i(\zeta), A_j(\nu)] = c_{ij}\, \delta'(\nu-\zeta) + \sum f_{ij}^q\, A_q(\zeta) \cdot \delta(\zeta-\nu)$ and therefore

$$(\zeta-\nu)^2 A_i(\zeta)\, A_j(\nu) = (\zeta-\nu)^2 A_j(\nu)\, A_i(\zeta). \tag{15}$$

Set $D(z_1,\dots,z_r) = \prod_{i<j} (z_i - z_j)^2$. It follows from (14) and (15) that the formal power series $\tilde{w}_r(z_1,\dots,z_r) := D(z_1,\dots,z_r)\, w_r(z_1,\dots,z_r)$ is S_r-invariant. Since $\lambda : U^c \to \mathbb{C}$ is trivial on $I_{n,k}^c$, (14) implies that the power series $w_r(z_1,\dots,z_r)$ does not contain z_r^m for $m < -n$. The same is true for $\tilde{w}_r(z_1,\dots,z_r)$, but since $\tilde{w}_r$ is S_r-invariant we see that for any $i \in \{1,\dots,r\}$ and $m < -n$, $\tilde{w}_r$ does not contain z_i^m. So we have proved that $w_r = f_r\, dz_1 \cdots dz_r$ where $f_r \in (\mathfrak{g}^*)^{\otimes r} \otimes K_r$ and that $w_0,\dots,w_k$ have the properties (1) and (2) from the definition of

$\Omega^c_{n,k}$. The property (3) follows from the "operator product expansion"

$$A_i(\zeta)\,A_j(\nu) = \frac{c_{ij}}{(\zeta-\nu)^2} + \sum_q f^q_{ij}\,\frac{A_q(\zeta)}{\zeta-\nu} + \cdots, \qquad |\zeta| \gg |\nu|. \tag{16}$$

Here $|\zeta| \gg |\nu|$ is just a heuristic way of saying that $(\zeta-\nu)^{-1} := \sum_{k=0}^{\infty} \zeta^{-k-1}\nu^k$, $(\zeta-\nu)^{-2} := \sum_{k=1}^{\infty} k\,\zeta^{-k-1}\nu^{k-1}$ and the dots in (16) denote a formal series $\sum_{i,j} a_{ij}\,\zeta^i\nu^j$ such that $a_{ij} \in U^c$ and for any $n \geq 0$ there is an M with the property that $a_{ij} \in I^c_n$ provided $i < -M$ or $j < -M$.

From the construction of $\psi : \Omega^c_{n,k} \to (U^c_k/I^c_{n,k})^*$ and $\psi^{-1} : (U^c_k/I^c_{n,k})^* \to \Omega^c_{n,k}$, it is clear that both mappings are continuous. ■

Remark. Our theorem has a global counterpart. In the case $c = 0$ it can be formulated as follows. Let G be an affine algebraic group over $\mathbb{C}$, X a connected smooth projective curve over $\mathbb{C}$, $\mathcal{F}$ a G-bundle on X such that $\operatorname{Aut}\mathcal{F}$ is finite, $S = \operatorname{Spec} B$ the base of the universal deformation of $\mathcal{F}$, and m the maximal ideal of B. Then B/m^k has the following description. Let $\mathfrak{g}_{\mathcal{F}}$ be the vector bundle on X corresponding to $\mathcal{F}$ and the adjoint representation of X. Let $\Omega_k(\mathcal{F})$ be the space of $(k+1)$-tuples $(w_0, \ldots, w_k)$ where w_r is a symmetric rational poly-differential on X^r with values in $\mathfrak{g}^*_{\mathcal{F}} \boxtimes \cdots \boxtimes \mathfrak{g}^*_{\mathcal{F}}$ having only simple poles at the diagonals $x_i = x_j$ with residues given by formula (4) for $c = 0$. Then B/m^k is canonically isomorphic to $\Omega_k(\mathcal{F})$. The proof will be given elsewhere.

Let us show that the isomorphism $\Omega^c_{n,k} \to (U^c_k/I^c_{n,k})^*$ is compatible with various structures on $\Omega^c_{n,k}$ and $(U^c_k/I^c_{n,k})^*$. First of all, the diagrams

$$\begin{array}{ccc} \Omega^c_{n,k} & \hookrightarrow & \Omega^c_{n+1,k} \\ \downarrow\wr & & \downarrow\wr \\ (U^c_k/I^c_{n,k})^* & \hookrightarrow & (U^c_k/I^c_{n+1,k})^* \end{array} \qquad \begin{array}{ccc} \Omega^c_{n,k+1} & \longrightarrow & \Omega^c_{n,k} \\ \downarrow\wr & & \downarrow\wr \\ (U^c_{k+1}/I^c_{n,k+1})^* & \longrightarrow & (U^c_k/I^c_{n,k})^* \end{array}$$

are commutative. Since the mapping $U^c_k/I^c_{n,k} \to U^c_{k+1}/I^c_{n,k+1}$ is injective, we obtain the following result.

Proposition 1. The mapping $\Omega^c_{n,k+1} \to \Omega^c_{n,k}$ is surjective.

Remarks. (1) The above proof of Proposition 1 makes use of formula (3) which follows from the Poincaré-Birkhoff-Witt theorem.

(2) Here is a sketch of a geometric proof of Proposition 1. Let $(w_0, \ldots, w_k) \in \Omega^c_{n,k}$. We must show that there is a $w_{k+1} \in (\mathfrak{g}^*)^{\otimes(k+1)} \otimes \Omega^K_{k+1}$ such that $(w_0, \ldots, w_{k+1}) \in \Omega^c_{n,k+1}$. Set $V = \operatorname{Spec}\mathbb{C}[[z_1, \ldots, z_{k+1}]]$. Let $\Delta_{ij} \subset V$ be the divisor $z_i = z_j$. Denote by Y the union of all subschemes of V of codimension 3 having the form $\Delta_{ij} \cap \Delta_{rs} \cap \Delta_{tu}$. Since $H^1(V\backslash Y, \mathcal{O}_V) = 0$, it is enough to show that for any $(z_1, \ldots, z_{k+1}) \in V\backslash Y$ there is an w_{k+1} which has the desired properties in a neighborhood of $(z_1, \ldots, z_k)$. There are two nontrivial cases: (1) $z_i = z_j$,

$z_r = z_s$, $i \neq j \neq r \neq s$, (2) $z_i = z_j = z_l$, $i \neq j \neq l$. In the second case the existence of w_{k+1} follows from the Jacobi identity in $\mathfrak{g}$.

Now consider the symbol epimorphism $(U_k^c/I_{n,k}^c)^* \to \mathrm{Sym}^k(\mathfrak{g}_K/(\mathfrak{g}\otimes\mathfrak{m}^n))$. It induces an injection $\Gamma^k((\mathfrak{g}_K/(\mathfrak{g}\otimes\mathfrak{m}^n))^*) \hookrightarrow (U_k^c/I_{n,k}^c)^*$ where Γ^k denotes the symmetric part of the k-tensor power. On the other hand we have a canonical isomorphism $(\mathfrak{g}_K/(\mathfrak{g}\otimes\mathfrak{m}^n))^* \xrightarrow{\sim} \mathfrak{g}^*\otimes\Omega_K^{(n)}$ where $\Omega_K^{(n)}$ is the space of differentials $\eta \in \Omega^K$ having a pole of order $\leq n$ at the point $z = 0$. It is easy to see that the diagram

$$\begin{array}{ccc} \Gamma^k((\mathfrak{g}_K/(\mathfrak{g}\otimes\mathfrak{m}^n))^*) & \hookrightarrow & (U_k^c/I_{n,k}^c)^* \\ \uparrow\wr & & \uparrow\wr \\ \Gamma^k(\mathfrak{g}^*\otimes\Omega_K^{(n)}) & \overset{f}{\hookrightarrow} & \Omega_{n,k}^c \end{array} \tag{17}$$

is commutative, where f is the linear mapping such that for any $\eta \in \mathfrak{g}^*\otimes\Omega_K^{(n)}$ one has $f(\eta^{\otimes k}) = (w_0, \ldots, w_k)$, $w_k(z_1, \ldots, z_k) = \eta(z_1)\otimes\cdots\otimes\eta(z_k)$, $w_r = 0$ for $r < k$.

Denote by V the space of invariant symmetric bilinear forms on $\mathfrak{g}$. If $c_1, c_2 \in V$, we have the comultiplication homomorphism $\Delta : U^{c_1+c_2} \to U^{c_1}\otimes U^{c_2}$ such that $\Delta(u) = u\otimes 1 + 1\otimes u$ for $u \in \mathfrak{g}_K \subset U^{c_1+c_2}$. It induces a mapping $U_k^{c_1+c_2}/I_{n,k}^{c_1+c_2} \to (U_k^{c_1}/I_{n,k}^{c_1})\otimes(U_k^{c_2}/I_{n,k}^{c_2})$ and therefore a mapping $(U_k^{c_1}/I_{n,k}^{c_1})^*\otimes(U_k^{c_2}/I_{n,k}^{c_2})^* \to (U_k^{c_1+c_2}/I_{n,k}^{c_1+c_2})^*$. So $\oplus_{c\in V}(U_k^c/I_{n,k}^c)^*$ becomes a V-graded commutative associative algebra with unit.

On the other hand, set $\Omega_k = \prod_{r=0}^k((\mathfrak{g}^*)^{\otimes r}\otimes\Omega_r^K)^{S_r}$; in other words, Ω_k is the space of $(k+1)$-tuples $(w_0, \ldots, w_k)$ where w_r is an S_r-invariant element of $(\mathfrak{g}^*)^{\otimes r}\otimes\Omega_r^K$. If $w', w'' \in \Omega_k$, $w' = (w'_0, \ldots, w'_k)$, $w'' = (w''_0, \ldots, w''_k)$, set $w'w'' = (w_0, \ldots, w_k)$ where

$$w_r = \sum_{i+j=r} \frac{1}{i!\,j!}\,\mathrm{Sym}\,(w'_i \boxtimes w''_j) \tag{18a}$$

$$(w'_i \boxtimes w''_j)(z_1, \ldots, z_{i+j}) = w'_i(z_1, \ldots, z_i)\otimes w''_j(z_{i+1}, \ldots, z_{i+j}) \tag{18b}$$

and Sym denotes the symmetrization operator (without the factor $1/r!$). Thus Ω_k becomes a commutative associative algebra with unit. Clearly $\Omega_{n,k}^c \subset \Omega_k$ and it is easy to see that $\Omega_{n,k}^{c_1}\cdot\Omega_{n,k}^{c_2} \subset \Omega_{n,k}^{c_1+c_2}$. The following result can be easily deduced from (5) or (12).

Proposition 2. The diagram

$$\begin{array}{ccc} (U_k^{c_1}/I_{n,k}^{c_1})^*\otimes(U_k^{c_2}/I_{n,k}^{c_2})^* & \longrightarrow & (U_k^{c_1+c_2}/I_{n,k}^{c_1+c_2})^* \\ \uparrow\wr & & \uparrow\wr \\ \Omega_{n,k}^{c_1}\otimes\Omega_{n,k}^{c_2} & \longrightarrow & \Omega_{n,k}^{c_1+c_2} \end{array}$$

is commutative.

Denote by Der K (resp. Der $\mathcal{O}$) the Lie algebra of continuous derivations of K (resp. of $\mathcal{O}$), i.e., the algebra of vector fields $f(z)d/dz$ where $f \in K$ (resp. $f \in \mathcal{O}$). The natural actions of Aut $\mathcal{O}$ and Der K on $\hat{\mathfrak{g}}^c_K$ induce the actions of Aut $\mathcal{O}$ and Der K on $(U^c_k/I^c_k)^* := \varinjlim_n (U^c_k/I^c_{n,k})^*$.

On the other hand, we have the natural actions of Aut $\mathcal{O}$ and Der K on Ω^K_r (change of variables and Lie derivative). Therefore Aut $\mathcal{O}$ and Der K act on $\Omega_k = \prod_{r=0}^k ((\mathfrak{g}^*)^{\otimes r} \otimes \Omega^K_r)^{S_r}$.

Proposition 3. (1) $\Omega^c_{n,k} \subset \Omega_k$ is invariant with respect to the action of Aut $\mathcal{O}$ and Der K, while $\Omega^c_{\infty,k} := \cup_{n=0}^{\infty} \Omega^c_{n,k}$ is invariant with respect to the action of Der K.

(2) The isomorphism $\Omega^c_{n,k} \xrightarrow{\sim} (U^c_k/I^c_{n,k})^*$ is equivariant with respect to Aut $\mathcal{O}$ and Der $\mathcal{O}$. The isomorphism $\Omega^c_{\infty,k} \xrightarrow{\sim} (U^c_k/I^c_k)^*$ is equivariant with respect to Aut $\mathcal{O}$ and Der K.

Proof. To prove statement (1), one has to show that $(z_i - z_2)^{-2} dz_1 dz_2 \in \Omega^K_2$ is Aut $\mathcal{O}$-invariant and Der K-invariant modulo polydifferentials regular at $z_1 = z_2$. To prove, e.g., Aut $\mathcal{O}$-invariance, we have to show that the expression

$$(z_1 - z_2)^{-2} dz_1 dz_2 - (\tilde{z}_1 - \tilde{z}_2)^{-2} d\tilde{z}_1 d\tilde{z}_2 \tag{19}$$

is regular for any change of variables $\tilde{z}_1 = \varphi(z_i)$. It is clear that (19) is symmetric with respect to z_1, z_2 and the order of the pole of (19) at $z_1 = z_2$ is not greater than 1. Therefore there is no pole at $z_1 = z_2$.

Statement (2) follows from (5). ∎

Let G be an algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. The adjoint actions of $G(K)$ and $\mathfrak{g}_K$ on $\mathfrak{g}_K$ induce the actions of $G(\mathcal{O})$ and $\mathfrak{g}_{\mathcal{O}} := \mathfrak{g} \otimes \mathcal{O}$ on $(U_k/I_{n,k})^*$ and also the actions of $G(K)$ and $\mathfrak{g}_K$ on $(U_k/I_k)^* := \varinjlim_n (U_k/I_{n,k})^*$. On the other hand $G(K)$ and $\mathfrak{g}_K$ act on $(\mathfrak{g}^*)^{\otimes r} \otimes \Omega^K_r$: if $g \in G(K)$, $a \in \mathfrak{g}_K$, $w \in \Omega^K_r$, then

$${}^g w(z_1, \ldots, z_r) = (\mathrm{Ad}_{g(z_1)} \otimes \cdots \otimes \mathrm{Ad}_{g(z_r)})(w(z_1, \ldots, z_r)) \tag{20}$$

$${}^a w(z_1, \ldots, z_r) = \sum_{i=1}^r (\mathrm{id}^{\otimes(i-1)} \otimes \mathrm{ad}_{a(z_i)} \otimes \mathrm{id}^{\otimes(r-i)})(w(z_1, \ldots, z_r)). \tag{21}$$

Let us explain that in (20) and (21) $\mathrm{Ad}_{g(z_1)}$ and $\mathrm{ad}_{a(z_i)}$ denote the operators $\mathfrak{g}^* \to \mathfrak{g}^*$ corresponding to $g(z_i)$ and $a(z_i)$ in the coadjoint representation while $\mathrm{id}^{\otimes(i-1)} \otimes \mathrm{ad}_{a(z_i)} \otimes \mathrm{id}^{\otimes(r-i)}$ is the operator $(\mathfrak{g}^*)^{\otimes r} \to (\mathfrak{g}^*)^{\otimes r}$ which acts as $\mathrm{ad}_{a(z_i)}$ on the ith tensor factor and identically on all the other ones. $G(K)$ and $\mathfrak{g}_K$ act on $\Omega_k = \prod_{r=0}^k ((\mathfrak{g}^*)^r \otimes \Omega^K_r)^{S_r}$ in the obvious way:

$${}^g(w_0, \ldots, w_k) = ({}^g w_0, \ldots, {}^g w_k), \qquad {}^a(w_0, \ldots, w_k) = ({}^a w_0, \ldots, {}^a w_k). \tag{22}$$

Proposition 4. (1) $\Omega_{n,k} \subset \Omega_k$ is invariant with respect to $G(\mathcal{O})$ and $\mathfrak{g}_{\mathcal{O}}$; $\Omega_{\infty,k} \subset \Omega_k$ is invariant with respect to $G(K)$ and $\mathfrak{g}_K$.

(2) The isomorphism $\Omega_{n,k} \xrightarrow{\sim} (U_k/I_{n,k})^*$ is equivariant with respect to $G(\mathcal{O})$ and $\mathfrak{g}_{\mathcal{O}}$. The isomorphism $\Omega_{\infty,k} \xrightarrow{\sim} (U_k/I_k)^*$ is equivariant with respect to $G(K)$ and $\mathfrak{g}_K$.

Proof. Statement (1) is obvious, while (2) follows from (5). ■

Now we are going to formulate the analog of Proposition 4 for an arbitrary c. The group $G(K)$ acts on $\hat{\mathfrak{g}}_K^c$ in the following way : if $g \in G(K), u \in \mathfrak{g}_K \subset \hat{\mathfrak{g}}_K^c$, then

$$ {}^g u = \mathrm{Ad}_g(u) + \operatorname*{res}_{z=0} c(u(z), g(z)^{-1} \cdot dg(z)) \cdot 1, \qquad {}^g 1 = 1 \tag{23} $$

where Ad denotes the adjoint action of $G(K)$ on $\mathfrak{g}_K$. Let us explain that if $g(z)$ is a G-valued function, then $g^{-1} \cdot dg$ (resp. $dg \cdot g^{-1}$) denotes the pullback with respect to g of the canonical left-invariant (resp. right-invariant) $\mathfrak{g}$-valued differential 1-form on G. The action of $G(K)$ on $\hat{\mathfrak{g}}_K^c$ defined by (23) and the adjoint action of $\mathfrak{g}_K$ on $\hat{\mathfrak{g}}_K^c$ induce the actions of $G(K)$ and $\mathfrak{g}_K$ on $(U_k^c/I_k)^*$.

Now let us introduce the *twisted actions* of $G(K)$ and $\mathfrak{g}_K$ on Ω_k in the following way: $g \in G(K)$ sends $w \in \Omega_k$ to

$$ w' = {}^g w \cdot \exp[-c \cdot dg \cdot g^{-1}] \tag{24} $$

while $a \in \mathfrak{g}_K$ sends $w \in \Omega_k$ to

$$ w'' = {}^a w - w \cdot [c \cdot da]. \tag{25} $$

Here we use the notation $[\eta] := (0, \eta, 0, \dots, 0) \in \Omega_K$ where $\eta \in \mathfrak{g}^* \otimes \Omega_K$ and c is considered as an operator $\mathfrak{g} \to \mathfrak{g}^*$. Let us explain that in (24)–(25) ${}^g w$ and ${}^a w$ are defined by (20)–(22), Ω_k is considered as an algebra with respect to the multiplication (18), and the exponent makes sense because $[\eta] \in \Omega_k$ is nilpotent for all $\eta \in \mathfrak{g}^* \otimes \Omega_K$. Here are the explicit formulae for w_1', w_2', w_1'', w_2'' in terms of w_0, w_1, w_2:

$$ w_1'(z) = \mathrm{Ad}_{g(z)} w_1(z) - w_0 \cdot c \cdot dg(z) \cdot g(z)^{-1} \tag{26} $$

$$ \begin{aligned} w_2'(z_1, z_2) = {} & (\mathrm{Ad}_{g(z_1)} \otimes \mathrm{Ad}_{g(z_2)})(w_2(z_1, z_2)) \\ & - w_1(z_1) \otimes c \cdot dg(z_2) \cdot g(z_2)^{-1} - c \cdot dg(z_1) \cdot g(z_1)^{-1} \otimes w_1(z_2) \\ & + w_0 \cdot c \cdot dg(z_1) \cdot g(z_1)^{-1} \otimes c \cdot dg(z_2) \cdot g(z_2)^{-1} \end{aligned} \tag{27} $$

$$ w_1''(z) = \mathrm{ad}_{a(z)} w_1(z_1) - w_0 \cdot c \cdot da(z) \tag{28} $$

$$w_2''(z) = (\mathrm{ad}_{a(z_1)} \otimes \mathrm{id})(w_2(z_1, z_2)) + (\mathrm{id} \otimes \mathrm{ad}_{a(z_2)})(w_2(z_1, z_2))$$
$$- w_1(z_1) \otimes c \cdot da(z_2) - c \cdot da(z_1) \otimes w_1(z_2) \tag{29}$$

Notice that (26) and (28) are essentially the usual gauge transformations.

The following proposition can be proved by direct computation.

Proposition 5. (1) $\Omega^c_{n,k} \subset \Omega_k$ is invariant with respect to the twisted action of $G(\mathcal{O})$ and $\mathfrak{g}_{\mathcal{O}}$ defined by (24)–(25). $\Omega_{\infty,k} \subset \Omega_k$ is invariant with respect to the twisted action of $G(K)$ and $\mathfrak{g}_K$.

(2) The isomorphism $\Omega^c_{n,k} \xrightarrow{\sim} (U^c_k/I^c_{n,k})^*$ is equivariant with respect to the twisted action of $G(\mathcal{O})$ and $\mathfrak{g}_{\mathcal{O}}$ on $\Omega^c_{n,k}$. The isomorphism $\Omega^c_{\infty,k} \xrightarrow{\sim} (U^c_k/I^c_k)^*$ is equivariant with respect to the twisted action of $G(K)$ and $\mathfrak{g}_K$ on $\Omega^c_{\infty,k}$.

Besides the adjoint action of $\hat{\mathfrak{g}}^c_K$ on $U(\hat{\mathfrak{g}}^c_K)^*$, there are two other natural actions: the "right" action of $a \in \hat{\mathfrak{g}}^c_K$ maps $\lambda \in U(\hat{\mathfrak{g}}^c_K)^*$ to $\lambda'(u) = \lambda(ua)$ while the "left" action of a maps λ to $\lambda''(u) = -\lambda(au)$. They induce the "right" and "left" actions of $\hat{\mathfrak{g}}^c_K$ on $\varprojlim_k \varinjlim_n (U^c_k/I^c_{n,k})^*$. Identifying $\varprojlim_k \varinjlim_n (U^c_k/I^c_{n,k})^*$ with $\varprojlim_k \Omega^c_{\infty,k}$ one obtains actions of $\hat{\mathfrak{g}}^c_K$ on $\varprojlim_k \Omega^c_{\infty,k}$ which will also be called "right" and "left." Of course $1 \in \hat{\mathfrak{g}}^c_K$ acts on $\varprojlim_k \Omega^c_{\infty,k}$ identically, so we only have to determine the action of $a \in \mathfrak{g}_K \subset \hat{\mathfrak{g}}^c_K$ on $\varprojlim_k \Omega^c_{\infty,k}$.

Proposition 6. The "right" (resp. "left") action of $a \in \mathfrak{g}_K \subset \hat{\mathfrak{g}}^c_K$ sends $w = (w_0, w_1, \ldots)$ $\in \varprojlim_k \Omega^c_{\infty,k}$ to $(\overline{w_0}, \overline{w_1}, \ldots)$ (resp. to $(\widetilde{w_0}, \widetilde{w_1}, \ldots)$) where

$$\overline{w_r}(z_1, \ldots, z_r) = \operatorname*{res}_{z_{r+1}=0} \eta_{r+1}(z_1, \ldots, z_{r+1}) \tag{30}$$

$$\widetilde{w_r}(z_1, \ldots, z_r) = -\overline{w_r}(z_1, \ldots, z_r) - \sum_{i=1}^{r} \operatorname*{res}_{z_{r+1}=z_i} \eta_{r+1}(z_1, \ldots, z_{r+1}) \tag{31}$$

and $\eta_{r+1}(z_1, \ldots, z_{r+1})$ is the polydifferential with values in $(\mathfrak{g}^*)^{\otimes r}$ obtained as a scalar product of $w_{r+1}(z_1, \ldots, z_{r+1})$ by $a(z_{r+1})$ with respect to the last tensor factor.

Proof. (30) follows immediately from (5). On the other hand, if $(\overline{w_0}, \overline{w_1}, \ldots)$ and $(\widetilde{w_0}, \widetilde{w_1}, \ldots)$ are respectively the results of the "right" and "left" action of $a \in \mathfrak{g}_K$ on w, then $(\overline{w_0}+\widetilde{w_0}, \overline{w_1}+\widetilde{w_1}, \ldots)$ is the result of the "adjoint" action of a on w. So according to Proposition 5 and formula (25) we have

$$\overline{w_r}(z_1, \ldots, z_r) + \widetilde{w_r}(z_1, \ldots, z_r) = \sum_{i=1}^{r} (\mathrm{id}^{\otimes(i-1)} \otimes \mathrm{ad}_{a(z_i)} \otimes \mathrm{id}^{\otimes(r-i)})(w_r(z_1, \ldots, z_r))$$
$$- \frac{1}{(r-1)!} \operatorname{Sym}(w_{r-1}(z_1, \ldots, z_{r-1}) \otimes c \cdot da(z_r))$$

where Sym has the same meaning as in (18a) and c is considered as an operator $\mathfrak{g} \to \mathfrak{g}^*$. This is equivalent to (31) by virtue of (4).

Here is another proof of (31). According to (5) we have to prove that if $\widetilde{w}_r$ is defined by (31), then

$$-\operatorname*{res}_{z_1=0} \cdots \operatorname*{res}_{z_r=0}(u_1(z_1) \otimes \cdots \otimes u_r(z_r), \widetilde{w}_r(z_1, \ldots, z_r)) = \operatorname*{res}_{z_1=0} \cdots \operatorname*{res}_{z_{r+1}=0}(a(z_1) \otimes u_1(z_2) \otimes \cdots \otimes u_r(z_{r+1}),\ w_{r+1}(z_1, \ldots, z_{r+1})) \tag{32}$$

for all $u_1, \ldots, u_r \in \mathfrak{g}_K$. The right-hand side of (32) is equal to $\operatorname{res}_{z_{r+1}=0}\operatorname{res}_{z_1=0} \cdots \operatorname{res}_{z_r=0}\xi(z_1, \ldots, z_{r+1})$ where $\xi(z_1, \ldots, z_{r+1}) = (u_1(z_1) \otimes \cdots \otimes u_r(z_r) \otimes a(z_{r+1}), w_{r+1}(z_1, \ldots, z_{r+1}))$. So (32) is equivalent to the formula $\operatorname{res}_{z_1=0} \cdots \operatorname{res}_{z_r=0}\operatorname{res}_{z_{r+1}=0}\xi(z_1, \ldots, z_{r+1}) + \sum_{i=1}^{r} \operatorname{res}_{z_1=0} \cdots \operatorname{res}_{z_r=0}\operatorname{res}_{z_{r+1}=z_i}\xi(z_1, \ldots, z_{r+1}) = \operatorname{res}_{z_{r+1}=0}\operatorname{res}_{z_1=0} \cdots \operatorname{res}_{z_r=0}\xi(z_1, \ldots, z_{r+1})$ which is easily deduced from Parshin's residue formula (10). ■

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References

[P] A. N. Parshin, *On the arithmetic of two-dimensional schemes*, *I. Distributions and residues*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), 736–773 (in Russian).

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